DISTRIBUTION FUNCTIONS

NOMENCLATURE & DEFINITIONS

1. DISTRIBUTION FUNCTION
   PHASE SPACE DENSITY $f(\mathbf{r}, \mathbf{v})$. Most often written as $f(\mathbf{v})$. Usually normalized such that $\int f(\mathbf{v}) d^3v = N$ (particles).
   Units of $f(\mathbf{r}, \mathbf{v})$ are $\frac{1}{\text{cm}^3 (\text{s}^3)}$ or $\frac{\text{particles}}{\text{cm}^3 (\text{s}^3)}$.
   Thus, $dN = f d^3v$

2. (Particle) Flux:
   $\mathbf{j} = \int \mathbf{v} f(\mathbf{v}) d^3v$
   Units of $\mathbf{j}$ are $\frac{\text{particles}}{\text{cm}^2 \text{s}}$.
   $d\mathbf{j} = \dot{\mathbf{v}} dN = \dot{\mathbf{v}} f d^3v$

   Differential (Particle) Flux:
   A. $\frac{d\mathbf{j}}{dE}$
   Units: $\frac{1}{\text{cm}^3 \text{s} \text{keV}}$ or $\frac{\text{particles}}{\text{cm}^3 \text{s} \text{keV}}$

   B. $\frac{d\mathbf{j}}{dE d\Omega}$
   Units: $\frac{1}{\text{cm}^3 \text{s} \text{keV ster}}$ or $\frac{\text{particles}}{\text{cm}^3 \text{s} \text{keV ster}}$

3. Energy Flux:
   $\mathbf{q} = \int \mathbf{v} dN \cdot E$
   Units: $\frac{\text{keV}}{\text{cm}^3 (\text{s}^3)} = \frac{\text{keV}}{\text{cm}^2 \text{s}}$
   $d\mathbf{q} = \dot{\mathbf{v}} dN \cdot \dot{E} f d^3v$

   Differential Energy Flux:
   A. $\frac{d\mathbf{q}}{dE}$
   Units: $\frac{\text{keV}}{\text{cm}^3 \text{s} \text{keV}}$

   B. $\frac{d\mathbf{q}}{dE d\Omega}$
   Units: $\frac{\text{keV}}{\text{cm}^3 \text{s} \text{keV ster}}$
From (eg. Maxwellian) to counts
\[ f(\mathbf{v}) = \frac{N}{\sqrt{2\pi} \mathbf{u}_e^2} e^{-\frac{(\mathbf{v} - \mathbf{u}_e)^2}{2\mathbf{u}_e^2}} \]
\[ \mathbf{u}_e = \sqrt{\frac{2kT}{m}} \]
\[ \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \]
\[ \text{Such that} \quad \int f dE' = N \]

Given \( N, \mathbf{u}_0, T \) can get \( f \) vs. \( v, \Theta, \phi \)
and \( f \) vs. \( E, \Theta, \phi \)

**Diff. Flux**
\[ \frac{d\tilde{J}}{dE d\Omega} \left( \text{cm}^2 \text{ s kev str} \right)^{-1} \]
\[ d\tilde{J} = \tilde{v} f(E) d^3v = \tilde{v} \int f(E) v^2 dv d\Omega = \tilde{v} \int f(E) v^2 dv d\Omega = \frac{\tilde{v} \int f(E) d\Omega d^3v}{dE} \]
\[ = \frac{2 \mathbf{u}_0^2}{m^2} \tilde{v} \int f(E) d\Omega d^3v \rightarrow \bar{E} f(E) d\Omega dE \]
\[ \frac{d\tilde{J}}{dE d\Omega} = \frac{2 \mathbf{u}_0^2}{m^2} \bar{E} f(E) \]
\[ \bar{E} = (E, \Theta, \phi) \]

Given \( N, \mathbf{u}_0, T \) can get \( d\tilde{J}/dE d\Omega \) vs. \( E, \Theta, \phi \)

**Diff. Energy Flux**
\[ \frac{d\bar{\mathbf{q}}}{dE d\Omega} \left( \text{kev} \text{s cm}^{-2} \text{ kev str} \right) \]
\[ d\bar{\mathbf{q}} = \tilde{v} E f(E) d^3v = \tilde{v} Ef(E) v^2 dv d\Omega = \tilde{v} Ef(E) v d(v^2) dv = \frac{\tilde{v} E f(E) dv}{dE} dE = \frac{2^2 E f(E) dv (v^2)}{m^2} \bar{E} f(E) dEd\Omega \]
\[ \frac{d\bar{\mathbf{q}}}{dE d\Omega} = \frac{2^2 E \bar{E} f(E)}{m^2} \bar{E} f(E) \]
4 Particle Count Rate

\[ \bar{R} \left( \text{s}^{-1} \right) = \frac{\frac{d\bar{\sigma}}{dE d\Omega}}{dE d\Omega} \cdot G \cdot E \]

\[ \Rightarrow \text{Geometric factor for electrostatic analyzers} \]

\[ G = \frac{\text{Area} \cdot \text{Solid Angle} \cdot \text{Bin Energy Width} \cdot \text{energy of channel}}{\text{cm}^2 \cdot \text{str} \cdot \text{eV} \cdot \text{eV}} \]

units: \( \text{cm}^2 \text{srMeV/eV} \)

Multiplied by \( \text{min} \) \( \frac{\frac{d\bar{\sigma}}{dE d\Omega} (\text{eV/cm}^2 \text{sr MeV})}{dE d\Omega} \) resulting in \( \text{#/s} \)

\( G \) is a function of energy channel, as well as detector direction. Thus \( G(E, \theta, \phi) \) depends on specifics of instrument.

\[ \Rightarrow \Delta E \text{ = energy band width} \]

For electrostatic analyzers the inherent energy resolution \( \Delta E \) is proportional to the energy: \( \Delta E = \frac{\theta}{2} \cdot \frac{R}{\Delta} \)

where \( \Delta \) is gap, \( R \) is radius. This is because \( \Delta E = q \sqrt{2} \cdot \frac{\Delta R}{\Delta} \) and \( \frac{\Delta E}{E} = \frac{\Delta R}{R} \cdot k \), where \( k \) is the analyzer constant. So \( \frac{\Delta E}{E} \) is constant and part of the geometric factor of the instrument.

\[ \Rightarrow E \text{ = detection efficiency} \]

This is the number of counts registered for a given number of particles that hit the detector. This gets us from \# particles/s to \# counts/s. Typical values for electrostatic analyzers is 0.6 - 0.7
5. Particle Counts:

\[ \bar{C} \text{ (unitless)} = \bar{E} \cdot \tau \]

\[ \tau \text{ is the measurement interval (seconds)} \]

\[ \text{Note: } \frac{\bar{R}}{G \cdot E} = \frac{d\bar{G}}{dE \cdot d\Omega} = \varepsilon \cdot \frac{dJ}{dE \cdot dS_2} \propto E^2 \cdot f \]

\[ \text{diff. energy flux} \quad \text{diff. flux} \quad \text{distr. function} \]

\[ \text{Units: } \bar{R} \left( \frac{\text{# counts}}{s} \right) \cdot \frac{d\bar{G}}{dE \cdot d\Omega} \left( \frac{\text{eV}}{\text{cm}^2 \cdot \text{s} \cdot \text{air}} \right) ; \quad G \left( \text{cm}^2 \cdot \text{str} \cdot \text{eV/eV} \right) \]

\[ \text{Note: counts/s are dead time corrected} \]

This dead-time correction is necessary before using the above equation.

Counts "C" is # counts per sample

\[ C = \bar{R} \cdot AT \], where \( AT \) is accumulation time.
DEAD TIME CORRECTIONS

In addition to geometric factor, detector efficiency and energy bandwidth, we need to apply a dead time correction to the counts measured to get the counts in the plasma. The dead time results from the processing that needs to take place once an event has been registered, as the DPU is occupied.

Thus the total dead time is a function of the number of counts:

\[ \text{Total DT} = C_m \cdot \text{EPT} \]

Live time (LT) is the time the detector was in operation:

\[ \text{LT} = \text{AT} - \text{DT} \]

where AT is the accumulation interval, or accumulation time. EPT = event processing time for 1 count.

The actual count rate is:

\[ R_{\text{measured}} = \frac{C_m}{\text{AT} - C_m \cdot \text{EPT}} \]

\[ R_{\text{measured}} = \frac{C_m}{\text{AT} - C_m \cdot \text{EPT}} = \frac{\text{C_m} / \text{AT}}{1 - \text{C_m} \cdot \text{EPT} / \text{AT}} \]

\[ 1 - R_{\text{measured}} \cdot \text{EPT} \]

Always < 1

In practice, you cannot have more particles than EPT’s.

Thus, dead time correction consists of dividing the measured count rate by the quantity \((1 - R_{\text{measured}} \cdot \text{EPT})\).

**Note:**\[ \frac{1}{\text{EPT}} = \text{max count rate} \] (because \[ \frac{1}{\text{EPT}} = \frac{\text{AT}}{\text{EPT} \cdot \text{AT}} = \frac{\text{max # of counts}}{\text{AT}} = \text{max count rate} \)]

For on-board processing on AMPTE/IRM the formula given on page 374 of D. Curtis's paper can be understood as follows:

For each sample

\[ R_{\text{real}} = \frac{C_m}{AT - C_m \cdot \text{EPT}} = \frac{C_m / \text{EPT}}{AT / \text{EPT} - C_m} \]

* The \( C_m / \text{EPT} \) in the numerator can be normalized away and re-entered on the ground. Thus \( CR_n(\theta) \) represents the normalized count rate = \( C_m / \text{EPT} \) in units that maximize the dynamical range of the on-board computations.

* \( (AT / \text{EPT}) \) is the accumulation time over a single energy measurement divided by the event processing time (in paper referred to as "dusting") and represents the maximum number of counts (denoted as \( CR_{\text{max}} \) in paper).

* \( C_m \) is the measured counts referred to as \( CR_i \) in the paper.

Thus

\[ R_{\text{real}} \rightarrow \frac{CR_n}{CR_{\text{max}} - CR_i} \]
FROM COUNTS TO DISTRIBUTION FUNCTION

\[
\text{Assuming } \frac{AE}{E} = \text{constant} = \frac{2AV}{V}
\]

\[
AT = \text{Accumulation time} = \frac{\text{Spin Period}}{16 \times 32} \text{s/sid}
\]

\[
EPT = \text{Event Processing Time} = 10^{-6} \text{s}
\]

\[
R_m = \text{measured count rate} = \frac{C_m}{AT}
\]

\[
C_m = \text{measured counts per sample}
\]

\[
R_r = \text{real count rate (slew time corrected)}
\]

\[
G = \text{Geometric factor (E)}
\]

\[
\Delta E = \text{Energy bandwidth (E)}
\]

\[
E = \text{Efficiency}(E)
\]

\[
R_r = \left( \frac{dJ}{de} \right) G . \frac{E}{E} = \frac{\phi}{\eta^2} . G . \frac{\Delta E}{E} . E = \frac{2E^2}{h^2} . f . G . \frac{360}{V} . E = \nu^4 f \sigma G \frac{\Delta E}{E}
\]

\[
\Rightarrow f(v) = \frac{C_m/AT}{1 - \frac{C_m}{AT} \cdot EPT} \cdot \frac{1}{V^4} \cdot \frac{1}{\left[ G \left( \frac{\Delta E}{E} \right) E \right]}
\]

Discrete Steps:

\[
f_{ijk} = \left( \frac{C_{ijk}/AT}{1 - C_{ijk}/AT \cdot EPT} \right) \cdot \frac{1}{V_i^4} \cdot \frac{1}{G_k \left( \frac{\Delta v}{V} \right) E_{ik}}
\]

\[
f_{ijk} = \frac{C_{ijk}/AT}{1 - C_{ijk}/AT \cdot EPT} \cdot \frac{1}{V_i^4} \cdot \frac{1}{G_k \cdot \text{eff.} \left( \frac{\Delta E}{E} \right)}
\]